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ON CERTAIN LOGICAL PARADOXES.

IN this paper I propose to discuss certain paradoxes, for the solution of which Mr. Whitehead and Mr. Russell have devised the "theory of types" which occupies so prominent a place in the first volume of *Principia Mathematica*. For the convenience of some of my readers I shall first give a brief account of these paradoxes; next I shall endeavor to show that the reasoning upon which the theory of types is based is fundamentally unsound; and finally I shall offer an alternative solution.

I shall not consider all of the paradoxes that are treated in *Principia Mathematica*, some of which belong to advanced parts of mathematics, but only the more fundamental examples that lie within the domain of logic itself.

I.

1. The first paradox to be noted arises from the consideration of a form of words in which a proposition appears to assert its own falsity. 'This is false' (where 'this' is supposed to refer to 'This is false' itself), and 'I am lying' are examples. If one of these be regarded as a significant statement, and the question is raised whether it is false or true, neither answer can be given without contradiction. If 'This is false' is true, then, since what it asserts is that it is false, it is false; and if it is false, then, since that is exactly what it asserts, it is true.

There are also expressions which seem to denote general propositions that are applicable to themselves, but which cannot be so applied without paradox. Suppose I write between parentheses: (Two and two is five. Green is the complementary of blue. All the propositions between these parentheses are false.) If this last expression is true, then, since it lies between the parentheses, it is false; and if it is false, then, since the other propositions between the parentheses are false, all the propositions are false—which is what the last one asserts, so that it is true.

2. A paradox of a somewhat different sort arises from the circumstance that there seem to be certain classes that include themselves as members. For example, there seems to be a class of all classes, and this—since it is itself a class—must be a member of itself; and there seems to be a class of all classes that contain more than one member, and this class would seem to have more than one member and so to be included in itself. However, if it seems hard to believe that a class can be a member of itself, it is easy to grant that there are classes that are *not* members of themselves. Let us consider the class of all such classes, *i. e.*, *the class of all classes that are not members of themselves;* and let us ask whether this class is a member of itself. If it is, then it belongs to the class of classes that are not members of themselves, so it too is not a member of itself; and if it is not, then it is precisely one of the classes which it includes as members.[1]

3. But the most important of the paradoxes to be considered arises from the notion of a property of a property. (Strange to say, this paradox is not explicitly mentioned in *Principia Mathematica*, although the solution that is offered applies most directly to it.) Can a property be a property of itself? It would seem, for example, that to be expressible in French is expressible in French. If this is correct, then, apparently, we may conceive of a certain property which a property may have, viz., the property of being a property of itself; and by contrast we may conceive of the property of *not* being a property of itself. If, now, we ask of this last property whether it is a property of

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[1] A more complex variation of this paradox arises from considering the notion of an ordered couple, the first member of which is a class of ordered couples that includes the couple in question as a member. (For example, the class of all ordered couples of similar things and the class of all ordered couples of contradictory propositions, themselves constitute an ordered couple of similar things; for they are similar in this respect at least, that if any couple belonging to either class is reversed it still belongs to the class.) By contrast let us consider the notion of an ordered couple whose first member is *not* a class of ordered couples which contains it as a member; and let the class of all such couples be called *K*. Then let us ask whether the ordered couple having *K* as its first member and (say) the center of the earth as its second member, belongs to the class of couples *K*. If it does, then its first member (*K*) does not contain it as a member. If it does not, then since its first member does not contain it as a member, it is just such a couple as belongs to *K*.

itself, we find that to answer yes is to say that it is not a property of itself, and that to answer no is to say that it is indeed a property of itself.[1]

II.

I do not intend to present a complete account of the solution of these paradoxes offered in *Principia Mathematica*. That will be found in the second chapter of the introduction to that work. I wish merely to examine the arguments upon which the solution is ultimately based, or, to change the figure, the first steps of the solution.[2]

It is asserted that no statement of the form, 'a has the property a,' can have a meaning;[3] and, more generally, that no statement of the form, 'a has the property b,' has a meaning if b presupposes a, *i. e.*, if b cannot be defined unless a already has a definite meaning. We shall have to examine the proof of this assertion.

Instead of the term 'property,' which I have here used, *Principia Mathematica* uses the term 'propositional function,' or simply 'function.' Let us consider some of the statements made with regard to the nature of functions. A certain difficulty arises in interpreting these statements, due to the fact that 'propositional function' is one of the indefinables of the system, and the authors do not hold themselves to the very strictest accountability with regard to their explanations of indefinable terms. They consider that such explanations form no essential part of their system and are of only 'practical' value in initiating the reader.

"By a 'propositional function' we mean something which contains a variable x, and expresses a *proposition* as soon as a value is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned" (p. 41).

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[1] Again a more complex variation is connected with the notion of a relation which obtains between itself and something else; as difference is different from identity. But no new principle is involved.

[2] At the outset a principle is announced, relating to what the authors call a certain class of vicious circles. This principle is: "Whatever involves all of a collection must not be one of the collection" (p. 40). In the order of proof, however, this general principle belongs at a relatively late stage; and I shall altogether disregard it here.

[3] This, if substantiated, directly disposes of the third paradox. The first and second are given a further elaborate treatment, into which it will not be necessary for us to enter.

The following comments suggest themselves:

1. The two statements quoted cannot both be true. According to the first, when a value is assigned to the variable which a function contains, it 'expresses' a proposition; according to the other, it *becomes* a proposition. Now is a function a mere formula; or is it, not a formula, but something which formulas are used to express? It appears to me obvious that the latter alternative is intended, and I shall for the moment assume it.

2. But the variable that the function is alleged to contain appears to be a *symbol*—that is to say, a symbol to which a variety of values may be assigned. Accordingly, the meaning underlying the contradictory statements above quoted would seem to be that *when a function is expressed in propositional form, the expression contains at least one variable.* And that is true.

3. The 'natural' form of expression for a function—the form our common languages have developed—is the infinitive phrase or the abstract noun: 'to be wise' or 'wisdom.' When it is expressed in propositional form—as 'x is wise,' or 'x has wisdom'—an ambiguity seems to arise, because this same form of words is also used to denote a proposition, and can only do so ambiguously. But, as denoting a function, the expression is entirely unambiguous.[1]

4. When a function is expressed in propositional form, it leaves gaps in the form that are filled up by variables. When a value is assigned to each of these variables, a proposition is denoted. This proposition, then, contains the function as one of its constituents, or factors.[2] The values assigned to the variables are likewise constituents of the proposition. It is convenient and proper to speak of a function of one, two, or

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[1] It should be observed that the use of the propositional form to denote functions is imperative in logic; because the infinitive phrase or abstract noun does not lend itself to the expression of the complicated ways in which functions of more than one variable may be combined in a single proposition.

[2] This is expressly denied in *Principia Mathematica* upon grounds which we shall consider below. It is said, for example, that "the function '$\hat{x}$ is human' is not a constituent of the proposition 'Socrates is human'" (p. 50).

more variables. But the function itself does not contain a variable.

5. In *Principia Mathematica* reference is made to "variable propositions," which, it is said, "it is necessary practically to distinguish" from functions (p. 41). The distinction appears to me to be a real one. Of course no proposition is a variable proposition. But, as was said above, an expression may ambiguously denote one of a number of propositions; and this in fact happens when the propositional expression of a function is used ambiguously to denote one of the propositions which contain that function.

Principia Mathematica continues: "It [the propositional function] agrees with the ordinary function of mathematics in the fact of containing an unassigned variable: where it differs is in the fact that the values of the function are propositions."

If the preceding comments are correct, this is absolutely wrong. As the propositional function contains no variable, so it is not itself a variable and has no 'values.'[1] Its expression in propositional form contains one or more variables; and when values are assigned to all these variables, a value is at the same time assigned to the whole expression, considered as denoting a proposition.

"It would seem . . . that the essential characteristic of a function is *ambiguity*. . . . When we speak of 'ϕx,' where x is not specified, we mean one value of the function, but not a definite one. We may express this by saying that 'ϕx' *ambiguously denotes* ϕa, ϕb, ϕc, etc., where ϕa, ϕb, ϕc, etc., are the various values of 'ϕx.'"

Here the function is once more identified with its expression in propositional form. It "denotes," and denotes "ambiguously." It would have been better to say: "A propositional function stands in a one–many relation to the propositions which contain it."

If it be said that I have misinterpreted the text, and that by 'propositional function' is always meant an expression, the facts of the matter are not altered. I shall then say that the same propositional function (the expression), which ambiguously denotes one of a number of propositions, *unambiguously denotes a property* which each of these propositions contains.

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[1] Later in this chapter the more acceptable expression, 'values for the function' is used instead of 'values of the function.'

We are now ready to consider the fundamental argument upon which the 'theory of logical types' rests. It is very brief and I shall quote it entire.

"When we say that 'ϕx' ambiguously denotes ϕa, ϕb, ϕc, etc., we mean that 'ϕx' means one of the objects ϕa, ϕb, ϕc, etc., though not a definite one, but an undetermined one. It follows that 'ϕx' only has a well-defined meaning (well-defined, that is to say, except in so far as it is of its essence to be ambiguous) if the objects ϕa, ϕb, ϕc, etc., are well-defined. That is to say, a function is not a well-defined function unless all its values are already well-defined. It follows from this that no function can have among its values anything which presupposes the function, for if it had, we could not regard the objects ambiguously denoted by the function as definite until the function was definite, while conversely, as we have just seen the function cannot be definite until its values are definite" (pp. 40, 41).[1]

In reply I would say: An expression of the sort denoted by 'ϕx' denotes a propositional function (*i. e.*, a property) with perfect unambiguity. It denotes a proposition ambiguously, irrespective of whether the various propositions which it might denote are all perfectly determined. In either case its meaning is well-defined if the constants that enter into it are well-defined. But in order that one of the propositions that are ambiguously denoted by the expression shall be well-defined, its peculiar terms (corresponding to the variables in the ambiguous expression) must also be well-defined.

But a remarkable comment upon the main argument remains to be considered: "It will be seen that according to the above account, the values of a function are presupposed by the function, not *vice versa*. It is sufficiently obvious, in any particular case, that the value of a function does not presuppose the function. Thus for example the proposition 'Socrates is human' can be perfectly apprehended without regarding it as a value of the function 'x is human.' It is true that, conversely, a function can be apprehended without its being necessary to apprehend its values severally and individually. If this were not the case no function could be apprehended at all, since the number of values (true or false) of a function is necessarily infinite and there are necessarily possible arguments [*i. e.*, values of the variables] with which we are unacquainted. What is necessary is not that the values should be given individually and extensionally, but that the totality of the values should be given intensionally, so that, concerning any assigned object, it is at least theoretically determinate whether or not the said object is a value of the function."

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[1] The passage continues: "This is a particular case, but perhaps the most fundamental one, of the vicious-circle principle."

Now, in the first place, it is to be observed that 'Socrates is human' cannot be understood by anyone who does not know what it is *to be human*. The statement, that to understand 'Socrates is human,' it need not be regarded as a value of the function 'x is human,' owes its plausibility to the fact that 'value,' 'function,' 'x,' are technical expressions, which may well enough not be understood by one who understands 'Socrates is human.'

In the second place, let us see what is implied in the amplified statement, that the totality of values must be given *intensionally*. For a totality to be given intensionally means that the propositional function which determines the totality shall be known. In this case, it means that the function, 'to be a value of the expression "ϕx,"' which I shall call $S(\phi x)$, must be known. Now suppose for the sake of argument that one might know this without knowing the function ϕx—though this seems to me a very unplausible supposition. Then in order to know the function $S(\phi x)$, the totality of its values must first be known; that is to say, known intensionally; which means that the function 'to be a value of $S(\phi x)$,' or $S[S(\phi x)]$ must be known; and so on *ad infinitum*. But this is ridiculous.

It seems to be unnecessary to carry the examination further.

III.

Let us now on our own account take up the study of the three paradoxes.

1. The first, which is, of course, the ancient 'Liar,' seems to me to be of very little significance, and I shall not dwell upon it. It turns upon the fact that the terms 'true' and 'false,' as applied to propositions may always be immediately eliminated. A proposition asserted to be true, may instead be simply asserted; and the contradictory of a proposition asserted to be false may be simply asserted. This use of 'true' and 'false' (as distinguished in particular from their application to beliefs) is in fact a mere periphrasis.

But when this operation is performed upon the phrase, 'This is true,' the term 'true' is not eliminated. For the proposition to be simply asserted is precisely that 'This is true.' The operation leaves the phrase as it was before; and no number of repetitions can alter the matter. The phrase, therefore, is meaningless.

The like may be said of the phrase 'This is false.' When the interpretation of the phrase by the elimination of 'false' is attempted, the result is, 'This (proposition above-mentioned) is true.' And when the operation is again attempted it yields only: 'This (proposition above-mentioned) is false.' This phrase also is therefore meaningless; and the question whether it is false or true, and with it the paradox of the 'Liar,' falls to the ground.[1]

The case of the general proposition denying itself is obviously similar: the terms 'true' and 'false' cannot be eliminated and hence are insignificant.

It should be noted that according to *Principia Mathematica* there cannot be a proposition which affirms anything with regard to itself. On its face, this does not seem to be accurate, and I see no sufficient reason for accepting it. 'This proposition is expressed in seven words' seems to be a proposition, and a true one. 'This proposition is expressed in German' seems to be a proposition, and a false one.[1] Furthermore, according to *Principia Mathematica*, a universal proposition about propositions, or about some kind of propositions, cannot be applied to itself. Thus there is no proposition, 'All propositions are equivalent to themselves,' from which it would follow that this proposition is itself equivalent to itself. I have not been able to discover the slightest reason for accepting this theory.

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[1] It is noteworthy that the same result follows from one of the most novel and suggestive theories of *Principia Mathematica*, a theory which, however, I am not quite ready to accept in its entirety. "A proposition is not a simple entity, but a relation of several; hence a statement in which a proposition appears as subject will be only significant if it can be reduced to a statement about the terms which appear in the proposition. A proposition, like such phrases as 'the so-and-so,' where grammatically it appears as the subject, must be broken up into its constituents if we are to find the true subject or subjects" (p. 51). It is clear that 'This is false' cannot be thus broken up, and hence, on this principle, is nonsense.

In this connection it may be remarked that if the theory just cited is correct, the long-standing controversy over the nature of implication is decided against Mr. Russell and his followers. According to these writers, the relation of 'material implication' is such that if P and Q are any two propositions, P must imply either Q or not-Q. Generally speaking, however, neither P nor Q can be eliminated from the phrase "P implies Q," which, accordingly, is to be regarded as nonsense.

2 and 3. The remaining paradoxes are of far greater interest and significance. It will be convenient for us to consider them to a certain extent together.

These paradoxes rest upon a common and very natural error with regard to the nature of the logical copula. It is assumed in the second paradox that there is a constant relation between any object and any class in which it is 'included.' This supposition has come to be universally made by logicians. A widely employed symbol for the alleged relation is Peano's ϵ. Similarly, in the third paradox it is assumed that there is a constant relation between any object and any property which it 'possesses.' To judge from my somewhat narrow reading, I should suppose that this supposition is not nearly so widely made. It is, however, closely connected with the foregoing. The relation between a property and the class of objects that it determines would seem to be constant; so that if the relation of the object to the class is constant, there would seem to be a constant relation of the object to the property. I shall use the symbol η to express this relation. Thus, as Peano would write: 'Horatius ϵ hero,' I shall write: "Horatius η heroism."

The difficulties involved in the assumption of the relation η are manifest. On the one hand it is a mere redundancy. To have a property is nothing else than the property itself. To have heroism is simply heroism. Accordingly, if the object a has the property ϕ, *Principia Mathematica* writes the proposition by simply placing ϕ and a side by side: $\phi\, a$—not in any such fashion as $a\ \eta\ \phi$. And this seems to me to be very sensible. On the other hand, if there is a relation η between object and property, it would seem that it is an important constituent in the proposition by which the property is affirmed of the object, and ought to be distinctly held in mind. But the relation η would itself be a property (or propositional function) of two variables. If η is necessary to connect ϕ and a, must not a further relation connect η with ϕ and a, and so on without limit? I do not see how an affirmative answer can be avoided. The consequence is that no proposition can ever be distinctly conceived.

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[1] On the theory mentioned in the last note, the proposition is in neither case the real subject, and can be rewritten so as to bring this fact out. Thus in the two examples given, the real subject may be said to be the language employed. But at any rate they seem to be real propositions, and not mere nonsense.

The relation η is, then, an illusion of language. What it is 'to have' a property depends upon the property.

But now the same thing seems to be equally true of the supposed relation ϵ, between an object and a class in which it is included. If ϕ is any property and ϕ' is the class which it determines, $\epsilon\phi'$ is equivalent to $\eta\phi$, and thus to ϕ itself, and hence is constant. But just as η has no fixed significance independent of ϕ, so—if I am right—ϵ has no fixed significance independent of ϕ' or, in the last resort, of ϕ. It no more denotes a relation than η does.

Let us see how this theory disposes of the logical paradoxes.

When anything stands in a relation to itself, we can express this by saying that the thing possesses a certain property, or belongs to a certain class—the class, namely, of things that have the property. Thus if Peter admires Peter, we may say that Peter has conceit, or is a conceited man; and if Paul admires Paul, we may say that Paul has conceit or is a conceited man; etc. Here a relation (*admires*), both of whose terms change from proposition to proposition, is replaced by the quasi-relation of possession of a property, or by that of membership in a class, the property or class remaining constant throughout. In symbols, $a\,\mathrm{R}\,a$, $b\,\mathrm{R}\,b$, $c\,\mathrm{R}\,c$ are replaced by $a\ \eta\ \phi$, $b\ \eta\ \phi$, $c\ \eta\ \phi$, where ϕ is a property, or by $a\ \epsilon\ \phi'$, $b\ \epsilon\ \phi'$, $c\ \epsilon\ \phi'$, where ϕ' is the class determined by the property ϕ.

This holds equally well when a class stands in a relation to itself. Thus let us suppose that among the clans of a certain savage tribe the relation R is found, where $x \mathrm{R} y$ means that the men of clan x are permitted to marry the women of clan y. Let us further suppose that in the case of certain clans, a, b, and c, the men are permitted to marry the women of the same clan. This would then be expressed by the formulas $a \mathrm{R} a$, $b \mathrm{R} b$, $c \mathrm{R} c$. Here, again, we may say that these clans possess a certain property, that of endogamy (ϕ), which determines a certain class of clans (ϕ'); and thus we may write $a \,\eta\, \phi$, $b \,\eta\, \phi$, $c \,\eta\, \phi$, or $a \epsilon \phi'$, $b \epsilon \phi'$, $c \epsilon \phi'$.

So also when a property stands in a relation to itself we may regard this as the possession of a property or as membership in the class thus defined. Thus the property of being a parent may be said to be reciprocal (R) to the property of being a child; and in the same sense the property of being a brother-or-sister, and the property of being a cousin, may be said to be reciprocal to themselves ($a \mathrm{R} a$, $b \mathrm{R} b$). These latter properties may accordingly be said to have the property of being self-reciprocal (ϕ), or to belong to the class of self-reciprocal properties (ϕ'); which would be symbolized by $a \,\eta\, \phi$, $b \,\eta\, \phi$, or $a \epsilon \phi'$, $b \epsilon \phi'$.

It is noteworthy, however, that in the examples which have just been cited, as in numberless others that suggest themselves, the relation R is, indeed, a genuine relation: it is not one of the quasi-relations, ϵ and η. The paradoxes which we are studying arise from the assumption that the same sort of restatement is possible in the case of ϵ and η themselves.

Taking first the case of ϵ, I see no reason to question that there may be propositions of the form $a \epsilon a$. The examples above cited—'The class of all classes is a class,' 'The class of all classes having more than one member has more than one member'—seem to me to be clear and true. The question is: Can these two propositions be reduced to the form, 'The class of all classes is a ϕ',' 'The class of classes having more than one member is a ϕ'?' Can we replace propositions symbolized by $a \epsilon a$, $b \epsilon b$, by the forms $a \epsilon \phi'$, $b \epsilon \phi'$?

Undoubtedly there is a temptation to assume that this transformation can be made. If I say: '*a* is a member of itself,' '*b* is a member of itself,' I seem to be attributing to *a* and *b* a common property[1] (that of being members of themselves) which marks them off as a distinct class. This, however, is an illusion. When I write $a \epsilon a$, $b \epsilon b$, the two expressions contain no common term, because ϵ has no fixed significance apart from *a* and *b*. Thus, while a given class may significantly be said to be included, or not to be included, in itself, there is no constant property of 'being included in itself,' or 'not being included in itself,' and no class of classes that include, or do not include, themselves. The paradox thus disappears.

The third paradox disappears in a precisely similar way. Even though we accept (as I do) the occurrence of true (not to say significant) propositions of the form $a \eta a$, $b \eta b$, it does not follow that these can be rewritten $a \eta \phi$, $b \eta \phi$. For η does not express a relation; and the two propositions, $a \eta a$ and $b \eta b$, have no common element. In the terminology of *Principia Mathematica*, there is no propositional function $x \eta x$.

I shall not delay over the more complicated forms of these paradoxes which are mentioned in the notes above. The solution turns upon the observation that just as there is no constant relation η, so there is no constant relation between all dual relations and their terms.

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[1] *Principia Mathematica* declares outright: "If such a supposition [viz., that some class is identical with one of its own members] had any meaning, '$a \epsilon a$' would be a significant propositional function."